Midterm - Differential Equations (2019-20)

Time: 2.5 hours. *Attempt all questions.*

The total marks is 27 but the maximum you can score is 25

1. Find the eigenvalues and eigenfunctions for $u''(t) + \lambda u(t) = 0$ on $[0, \pi]$ with the endpoint conditions $u'(0) = u(\pi) = 0$. **[4 marks]**

2. Find the general solution $u(t)$ of the differential equation $u'' + 2u' + 2u = e^{-t}$. **[4 marks]**

3. Give an example to show that non-uniqueness might hold for differential equations of the form $u'(t) = f\big(u(t)\big)$ with continuous $f : \mathbf{R} \to \mathbf{R}$, unless some additional assumptions are made on f. **[3 marks]**

4. Find the general solution $y(x)$ (in the form of a power series) to the differential equation $y'' + xy = 0$. **[4 marks]**

5. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuously differentiable function. Consider the differential equation $u'(t) = f\big(u(t)\big)$ with initial profile $u(0) = u_0$ and suppose that $f(u_0) < 0$. Show that the solution $u(t)$ converges to the first zero of f to the left of u_0 (if one such zero exists). If there is no such zero show that the solution converges to $-\infty$. **[5 marks]**

6. Consider the differential equation $u'(t) = f\big(u(t)\big)$, where $f : \mathbf{R} \to \mathbf{R}$ is Lipschitz.

   (a) Show that for any $t_0 \in \mathbf{R}$, $u(t)$ is a solution if and only if $u(t - t_0)$ is a solution. **[2 marks]**

   (b) Show that for two maximal solutions $\{x_j(t),\ t \in I_j\}$, $j = 1, 2$, the images $\gamma_j = \{x_j(t) : t \in I_j\}$ either coincide or are disjoint. **[5 marks]**